Retake test 3 Numerical Mathematics 2
February, 2022

Duration: one hour.

In front of the questions one finds the points. The sum of the points plus 1 gives the end mark of this test. Use of a calculator is allowed.

1. Consider the inner product $(f, g) = \int_0^\infty \exp(-3x)f(x)g(x)dx$.

   (a) [2.5] Derive the first two orthogonal polynomials (so up to a quadratic polynomial)associated with this inner product.

   (b) [1.5] Which integral related to the above inner product can be approximated by a Gauss rule based on the orthogonal polynomials of part a? Use the linear orthogonal polynomial of the previous part to determine the associated Gauss rule.

   (c) [0.5] According to the theory, how does the 'degree of exactness' of a Gauss rule relate to the degree of the orthogonal polynomial used to define the rule.

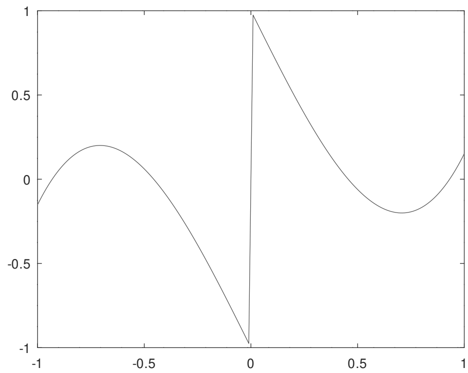
2. Consider the function $f(x)$ on [-1,1] given by

$$f(x) = \begin{cases} -1 & \text{for } x \in [-1, 0], \\ 1 & \text{for } x \in (0, 1]. \end{cases}$$

   (a) [1] Let $C_n(x) = \sum_{k=0}^{n} a_k T_k(x)$ be the Chebyshev expansion of $f(x)$. Give the formula for a_k and use this to show $a_k = 0$ for k even.

   (b) [2.5] Show that $a_k = \frac{4}{k\pi}(-1)^{\frac{k-1}{2}}$ for k odd. Next determine $C_3(x)$.
   For this exercise you might need $\cos(2\theta) = 2\cos^2(\theta) - 1$, $\cos(3\theta) = 4\cos^3(\theta) - 3\cos(\theta)$ or $\sin(3\theta) = 3\sin(\theta) - 4\sin^3(\theta)$.

   (c) [0.5] In the figure below the error of the approximation is shown for $n = 4$. Why is the error an odd function?
   Despite that $f(x)$ is not continuous, apply the the theorem of de la Vallée-Poussin to derive an as sharp as possible lower bound for the the minimax error?



   (d) [0.5] Will $C_n(x)$ converge to $f(x)$ in the norm associated to the inner product that defines the Chebyshev polynomials for n tending to infinity? Will $C_n(x)$ converge pointwise to $f(x)$ on the whole interval [-1,1] for n tending to infinity?